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THE BITANGENTIAL OF THE QUINTIC.

By Wm. E. Heal, Marion, Indiana.

The problem of finding a curve which shall pass through the points of contact of the bitangents of a given curve has engaged the attention of the most eminent mathematicians. Salmon, in his Higher Plane Curves, gives two methods of finding the equation of the bitangential curve, but the application of these methods is attended with difficulties which have not been overcome except for the single case of the curve of the fourth order. The equation of the bitangential of the quartic is given by Salmon in a very elegant form, but in attempting to apply the same methods to the quintic he was not successful. He says (Higher Plane Curves, 3d edition, p. 351), "In order to form the bitangential curve of a quintic the quantity to be calculated is

$$(27Q_2Q_5 - 5Q_3Q_4)^2 = 5(4Q_3^2 - 9Q_2Q_4)(5Q_4^2 - 12Q_3Q_5)$$

a quantity containing $\alpha\beta\gamma$ in the sixth order, and which it is necessary, by the help of the equation of the curve, to show to be divisible by R^6. Now, in virtue of a formula already obtained, we have

$$4Q_3^2 - 9Q_2Q_4 = R^2(4\theta - 9H\Phi).$$

It is also easy to show that $27Q_2Q_5 - 5Q_3Q_4$ and $5Q_4^2 - 12Q_3Q_5$ are each divisible by R; but I have not been able to carry the reduction further."

And again (p. 357, foot note): "I attempted in like manner to obtain the bitangential curve of a quintic by writing down for the curve, whose equation is given Art. 394, a covariant of the right order and such that the absolute term vanished if the axis of x touches the given curve a second time. For instance, if $\psi = 4\theta - 9H\Phi$, then $A\left(\frac{d\psi}{dx}\right)^2 + \ldots$ and $\psi\left[A\frac{d^2\psi}{dx^2} + \ldots\right]$ are covariants of the right order. Although I have not been successful, it may be useful for purposes of reference to give the values I obtained for the covariants in this case.

It will be seen that, without loss of generality, we may suppose c_1 and c_2 to vanish. We have, then,

$$\begin{aligned} H = \; & b^2c + 3b^2(d_0x + d_1y) \\ & + 3(b^2c_0 - 4bcd_1)x^2 + 3(2b^2c_1 - 5bcd_2)xy \\ & + 3(b^2c_2 - bcd_3)y^2 + (b^2f_0 - 16bce_1 + 18c^2d_2)x^3 \\ & + (3b^2f_1 - 39bce_2 - 9bd_0d_2 + 9bd_1^2 + 18c^2d_3)x^2y \\ & + (-6bcf_1 - 12bd_0e_1 + 12be_0d_1 + 18c^2e_2 + 24cd_0d_2 - 18cd_1^2)x^4 \\ & + \ldots, \end{aligned}$$

$$\Theta = 9b^2 \left\{ \begin{array}{l} b^4d_0^2 + 6b^3c^2d_1 \\ + \; (4b^4d_0e_0 + 12b^3c^2e_1 - 6b^3cd_0d_1 - 57b^2c^3d_2)x \\ + \left\{ \begin{array}{l} 2b^4d_0f_0 + 4b^4e_0^2 + 6b^3c^2f_1 + 6b^3cd_0e_1 - 48b^3cd_1e_0 \\ - 105b^2c^3e_2 - 293b^2c^2d_0d_2 + 269b^2c^2d_1^2 + 36bc^4d_3 \end{array} \right\} x^2 \\ + \ldots \end{array} \right\},$$

$$\Phi = 6b \left\{ \begin{array}{l} b^3e_0 + 4b^2cd_1 + (b^3f_0 - 8b^2ce_1 - 38bc^2d_2)x \\ + [b^3f_1 - 2b^2ce_2 + 27b^2(d_1^2 - d_0d_2) - 41bc^2d_3]y \\ + \left\{ \begin{array}{l} - 12b^2cf_1 - 12b^2d_0e_1 + 12b^2e_0d_1 + 6bc^2e_2 \\ - 162bcd_0d_2 + 168bcd_1^2 - 6c^3d_3 \end{array} \right\} x^2 \\ + \ldots \end{array} \right\}.$$

Of the quantities A, B, etc., the only ones which contain terms independent of x and y are $A = b^2$, $F = bc$; so that if any quantity ψ of the form $\Theta + kH\Phi$ written at full length be

$$A + B_0x + B_1y + C_0x^2 + \ldots,$$

then the degree of ψ being 22, the absolute term in the covariant $A\left(\frac{d\psi}{dx}\right)^2 + \ldots$ is $b^2B_0^2 + 44bcAB_1$, and in $A\frac{d^2\psi}{dx^2} + \ldots$ is $2b^2C_0 + 42bcB_1$."

I quote thus in full for the reason that I have recently discovered the functions sought and have verified the result by the method indicated by Salmon.

I gave a form for the bitangential of a quintic in the Analyst, Vol. VIII, p. 171. The equation was obtained by analogy from the equation of the bitangential of the quartic; but I stated that I had not verified the result, and the results of the present paper show that it was not the true form.

Let us take the origin on the curve, and the axis of x as the tangent. The equation of the curve may then be written in the form

$$\begin{aligned} u = 5by &+ 10(c_0x^2 + 2c_1xy + c_2y^2) \\ &+ 10(d_0x^3 + 3d_1x^2y + 3d_2xy^2 + d_3y^3) \\ &+ \ 5(e_0x^4 + 4e_1x^3y + 6e_2x^2y^2 + 4e_3xy^3 + e_4y^4) \\ &+ (f_0x^5 + 5f_1x^4y + 10f_2x^3y^2 + 10f_3x^2y^3 + 5f_4xy^4 + f_5y^5) \\ = 0. & \end{aligned}$$

I prefer to work with this form, which is more general, than with that used by Salmon, although my results were first obtained by the use of his equations given in the foot note before mentioned.

The points in which the axis of x meets the curve are found by making $y = 0$ in this equation, and are therefore given by the equation

$$x^2(f_0x^3 + 5e_0x^2 + 10d_0x + 10c_0) = 0.$$

Two of these points coincide with the origin, as they evidently should, the axis being a tangent. Two other points in which the axis meets the curve will coincide, or the axis will be a double tangent if the discriminant of

$$f_0x^3 + 5e_0x^2 + 10d_0x + 10c_0 = 0$$

vanishes; that is, if we have

$$27c_0^2f_0^2 + 40d_0^3f_0 + 50c_0e_0^3 - 25d_0^2e_0^2 - 90c_0d_0e_0f_0 = 0.$$

We now write for the second differential coefficients of u

$$a = \frac{d^2u}{dx^2} = 20\left\{\begin{array}{l} c_0 + 3(d_0x + d_1y) + 3(e_0x^2 + 2e_1xy + e_2y^2) \\ \quad + (f_0x^3 + 3f_1x^2y + 3f_2xy^2 + f_3y^3) + \dots \end{array}\right\},$$

$$b = \frac{d^2u}{dy^2} = 20\left\{\begin{array}{l} c_2 + 3(d_2x + d_3y) + 3(e_2x^2 + 2e_3xy + e_4y^2) \\ \quad + (f_2x^3 + 3f_3x^2y + 3f_4xy^2 + f_5y^3) + \dots \end{array}\right\},$$

$$c = \frac{d^2u}{dz^2} = 20\left\{\begin{array}{l} 3by + 3(c_0x^2 + 2c_1xy + c_2y^2) \\ \quad + (d_0x^3 + 3d_1x^2y + 3d_2xy^2 + d_3y^3) + \dots \end{array}\right\},$$

$$f = \frac{d^2u}{dydz} = 20\left\{\begin{array}{l} b + 3(c_1x + c_2y) + 3(d_1x^2 + 2d_2xy + d_3y^2) \\ \quad + (e_1x^3 + 3e_2x^2y + 3e_3xy^2 + e_4y^3) + \dots \end{array}\right\},$$

$$g = \frac{d^2u}{dzdx} = 20\left\{\begin{array}{l} 3(c_0x + c_1y) + 3(d_0x^2 + 2d_1xy + d_2y^2) \\ \quad + (e_0x^3 + 3e_1x^2y + 3e_2xy^2 + e_3y^3) + \dots \end{array}\right\},$$

$$h = \frac{d^2u}{dxdy} = 20 \begin{Bmatrix} c_1 + 3(d_1x + d_2y) + 3(e_1x^2 + 2e_2xy + e_3y^2) \\ + (f_1x^3 + 3f_2x^2y + 3f_3xy^2 + f_4y^3) + \ldots \end{Bmatrix}.$$

If we now write

$$\Sigma = \begin{vmatrix} a, & h, & g, & \alpha \\ h, & b, & f, & \beta \\ g, & f, & c, & \gamma \\ \alpha, & \beta, & \gamma, & 0 \end{vmatrix},$$

we have

$$\Sigma = A\alpha^2 + B\beta^2 + C\gamma^2 + 2F\beta\gamma + 2G\gamma\alpha + 2H\alpha\beta,$$

where

$$\begin{aligned} A = bc - f^2 & \\ = - b^2 - 6bc_1x - 3bc_2y + 3(c_0c_2 - 2bd_1 - 3c_1^2)x^2 & \\ - 3(bd_2 + 4c_1c_2)xy + 3(bd_3 - 2c_2^2)y^2 & \\ + (c_2d_0 - 2be_1 + 9c_0d_2 - 9c_1d_1)x^3 & \\ + 3(be_2 + 3c_0d_3 - 5c_2d_1 - 6c_1d_2)x^2y & \\ + 12(be_3 - 2c_2d_2)xy^2 + (7be_4 - 8c_2d_3)y^3 & \\ + \ldots, & \end{aligned}$$

$$B = ac - g^2 = 3bc_0y + \ldots,$$

$$\begin{aligned} C = ab - h^2 = (c_0c_2 - c_1^2) + 3(c_0d_2 + c_2d_0 - 2c_1d_1)x & \\ + 3(c_0d_3 + c_2d_1 - 2c_1d_2)y + \ldots, & \end{aligned}$$

$$F = gh - af = - bc_0 - 3bd_0x + 3(c_1^2 - c_0c_2 - bd_1)y + \ldots,$$

$$\begin{aligned} G = hf - bg = bc_1 + 3(c_1^2 + bd_1 - c_0c_2)x + 3bd_2y & \\ + 3(be_1 - c_2d_0 - 3c_0d_2 + 4c_1d_1)x^2 & \\ + 3(c_2d_1 + 2be_2 + 2c_1d_2 - 3c_0d_3)xy & \\ + 3(be_3 - 2c_1d_3 + 2c_2d_2)y^2 & \\ + (bf_1 - c_2e_0 + 9d_1^2 - 9c_0e_2 - 9d_0d_2 + 10c_1e_1)x^3 & \\ + 3(bf_2 + 2c_2e_1 + 3d_1d_2 - 3d_0d_3 + 4c_1e_2 - 6c_0e_3)x^2y & \\ + 3(bf_3 - 2c_1e_3 + 3d_2^2 - 3d_1d_3 - 3c_0e_4 + 5c_2e_2)xy^2 & \\ + (bf_4 - 8c_1e_4 + 8c_2e_3)y^3 & \\ + \ldots, & \end{aligned}$$

$$
\begin{aligned}
H = fg - ch = {} & 3bc_0x + 3(bd_0 + 2c_0c_1)x^2 \\
& + 3(c_1^2 - bd_1 + 3c_0c_2)\,xy + 6(c_1c_2 - bd_2)y^2 \\
& + (be_0 + 8c_1d_0)x^3 \\
& + 3(2c_1d_1 - 2be_1 + 3c_0d_2 + 3c_2d_0)x^2y \\
& + 3(2c_1d_2 + 3c_0d_3 + 3c_2d_1 - 5be_2)xy^2 \\
& + 8(c_1d_3 - be_3)y^3 \\
& + \ldots,
\end{aligned}
$$

the Hessian is

$$
H = \begin{vmatrix} a, & h, & g \\ h, & b, & f \\ g, & f, & c \end{vmatrix}
= aA + hH + gG.
$$

Introducing the values of a, h, g, A, H, G we have for the equation of the Hessian

$$
\begin{aligned}
H = {} & b^2c_0 + 3b^2d_0x + 3(bc_0c_2 - bc_1^2 + b^2d_1)y \\
& + 3(b^2e_0 + 2c_0^2c_2 - 2c_0c_1^2 + 4bc_1d_0 - 4bc_0d_1)x^2 \\
& + 3(2bc_1d_1 + 2b^2e_1 + 3bc_2d_0 + 4c_0c_1c_2 - 4c_1^3 - 5bc_0d_2)xy \\
& + 3(b^2e_2 - bc_0d_3 + 2c_0c_2^2 - 2c_1^2c_2 - 2bc_1d_2 + 3bc_2d_1)y^2 \\
& + (b^2f_0 + 8c_0c_2d_0 + 10c_1^2d_0 + 16bc_1e_0 - 16bc_0e_1 + 18c_0^2d_2 - 36c_0c_1d_1)x^3 \\
& + 3\left\{\begin{aligned} & b^2f_1 + 3bd_1^2 + 3bc_2e_0 - 3bd_0d_2 - 4c_0c_2d_1 \\ & + 6c_0^2d_3 + 10bc_1e_1 + 12c_1c_2d_0 - 13bc_0e_2 - 14c_1^2d_1 \end{aligned}\right\} x^2y \\
& + 3\left\{\begin{aligned} & b^2f_2 - bc_1e_2 + 2c_0c_2d_2 + 3bd_1d_2 - 3bd_0d_3 \\ & + 6c_0c_1d_3 + 6c_2^2d_0 + 6bc_2e_1 - 7bc_0e_3 - 14c_1^2d_2 \end{aligned}\right\} xy^2 \\
& + \left\{\begin{aligned} & b^2f_3 + 7bc_1e_3 - 7bc_0e_4 - 8c_1^2d_3 + 8c_0c_2d_3 + 9bd_2^2 \\ & + 9bc_2e_2 - 9bd_1d_3 + 18c_2^2d_1 - 18c_1c_2d_2 \end{aligned}\right\} y^3 \\
& + \ldots.
\end{aligned}
$$

If we now write for α, β, γ the differential coefficients of H with respect to x, y, z in the expression for Σ, we have the covariant Θ. Thus

$$\Theta = \begin{vmatrix} a, & h, & g, & \dfrac{dH}{dx} \\ h, & b, & f, & \dfrac{dH}{dy} \\ g, & f, & c, & \dfrac{dH}{dz} \\ \dfrac{dH}{dx}, & \dfrac{dH}{dy}, & \dfrac{dH}{dz}, & 0 \end{vmatrix}$$

$$= A\left(\frac{dH}{dx}\right)^2 + B\left(\frac{dH}{dy}\right)^2 + C\left(\frac{dH}{dz}\right)^2$$

$$+ 2F\left(\frac{dH}{dy}\right)\left(\frac{dH}{dz}\right) + 2G\left(\frac{dH}{dz}\right)\left(\frac{dH}{dx}\right) + 2H\left(\frac{dH}{dx}\right)\left(\frac{dH}{dy}\right).$$

We easily find from the value given for H

$$\begin{aligned} \frac{dH}{dx} &= 3b^2d_0 + 6(b^2e_0 + 2c_0^2c_2 - 2c_0c_1^2 + 4bc_1d_0 - 4bc_0d_1)x \\ &+ 3(2bc_1d_1 + 2b^2e_1 + 3bc_2d_0 + 4c_0c_1c_2 - 4c_1^3 - 5bc_0d_2)y \\ &+ 3(b^2f_0 + 8c_0c_2d_0 + 10c_1^2d_0 + 16bc_1e_1 - 16bc_0e_1 + 18c_0^2d_2 - 36c_0c_1d_1)x^2 \\ &+ 6\begin{Bmatrix} b^2f_1 + 3bd_1^2 + 3bc_2e_0 - 3bd_0d_2 - 4c_0c_2d_1 + 6c_0^2d_3 \\ + 10bc_1e_1 + 12c_1c_2d_0 - 13bc_0e_2 - 14c_1^2d_1 \end{Bmatrix} xy \\ &+ \ldots, \end{aligned}$$

$$\begin{aligned} \frac{dH}{dy} &= 3(bc_0c_2 - bc_1^2 + b^2d_1) + 3(2bc_1d_1 + 2b^2e_1 + 3bc_2d_0 + 4c_0c_1c_2 - 4c_1^3 - 5bc_0d_2)x \\ &+ 6(b^2e_2 - bc_0d_3 + 2c_0c_2^2 - 2c_1^2c_2 - 2bc_1d_2 + 3bc_2d_1)y \\ &+ 6\begin{Bmatrix} b^2f_2 - bc_1e_2 + 2c_0c_2d_2 + 3bd_1d_2 - 3bd_0d_3 + 6c_0c_1d_3 \\ + 6c_2^2d_0 + 6bc_2e_1 - 7bc_0e_3 - 14c_1^2d_2 \end{Bmatrix} xy \\ &+ \ldots, \end{aligned}$$

$$\begin{aligned} \frac{dH}{dz} &= 9b^2c_0 + 24b^2d_0x + 24(bc_0c_2 - bc_1^2 + b^2d_1)y \\ &+ 21(b^2e_0 + 2c_0^2c_2 - 2c_0c_1^2 + 4bc_1d_0 - 4bc_0d_1)x^2 \\ &+ 21(2bc_1d_1 + 2b^2e_1 + 3bc_2d_0 + 4c_0c_1c_2 - 4c_1^3 - 5bc_0d_2)xy \\ &+ \ldots. \end{aligned}$$

Whence the value of θ is

$$\theta = (9b^6d_0^2 + 27b^4c_0^2c_1^2 - 27b^4c_0^3c_2 + 54b^5c_0^2d_1 - 54b^5c_0c_1d_0)$$
$$+ \left\{\begin{array}{l} - 27b^4c_0^2c_2d_0 + 36b^6d_0e_0 + 54b^5c_1d_0^2 - 54b^5c_0d_0d_1 + 108b^5c_0^2e_1 \\ - 108b^5c_0c_1e_0 - 486b^4c_0c_1^2d_0 - 513b^4c_0^3d_2 + 1026b^4c_0^2c_1d_1 \end{array}\right\} x$$
$$+ \ldots .$$

Also, if we write differential symbols for α, β, γ in the expression for Σ, and operate on H, we get the covariant φ. Thus

$$\Phi = \begin{vmatrix} a, & h, & g, & \dfrac{d}{dx} \\ h, & b, & f, & \dfrac{d}{dy} \\ g, & f, & c, & \dfrac{d}{dz} \\ \dfrac{d}{dx}, & \dfrac{d}{dy}, & \dfrac{d}{dz}, & 0 \end{vmatrix} H$$

$$= A\frac{d^2H}{dx^2} + B\frac{d^2H}{dy^2} + C\frac{d^2H}{dz^2} + 2F\frac{d^2H}{dydz} + 2G\frac{d^2H}{dzdx} + 2H\frac{d^2H}{dxdy}.$$

We easily find

$$\frac{d^2H}{dx^2} = 6(b^2e_0 + 2c_0^2c_2 - 2c_0c_1^2 + 4bc_1d_0 - 4bc_0d_1)$$
$$+ 6\left\{\begin{array}{l} b^2f_0 + 8c_0c_2d_0 + 10c_1^2d_0 + 16bc_1e_1 \\ - 16bc_0e_1 + 18c_0^2d_2 - 36c_0c_1d_1 \end{array}\right\} x$$
$$+ \ldots,$$

$$\frac{d^2H}{dxdy} = 3(2bc_1d_1 + 2b^2e_1 + 3bc_2d_0 + 4c_0c_1c_2 - 4c_1^3 - 5bc_0d_2)$$
$$+ 6\left\{\begin{array}{l} b^2f_1 + 3bd_1^2 + 3bc_2e_0 - 3bd_0d_2 - 4c_0c_2d_1 + 6c_0^2d_3 \\ + 10bc_1e_1 + 12c_1c_2d_0 - 13bc_0e_2 - 14c_1^2d_1 \end{array}\right\} x$$
$$+ \ldots,$$

$$\frac{d^2H}{dxdz} = 24b^2d_0 + 42(b^2e_0 + 2c_0^2c_2 - 2c_0c_1^2 + 4bc_1d_0 - 4bc_0d_1)x$$
$$+ \ldots,$$

$$\frac{d^2H}{dy^2} = 6(b^2e_2 - bc_0d_3 + 2c_0c_2^2 - 2c_1^2c_2 - 2bc_1d_2 + 3bc_2d_1)$$
$$+ 6\begin{Bmatrix} b^2f_2 - bc_1e_2 + 2c_0c_2d_2 + 3bd_1d_2 - 3bd_0d_3 + 6c_0c_1d_3 \\ + 6c_2^2d_0 + 6bc_2e_1 - 7bc_0e_3 - 14c_1^2d_2 \end{Bmatrix} x$$
$$+ \ldots,$$

$$\frac{d^2H}{dydz} = 24(bc_0c_2 - bc_1^2 + b^2d_1)$$
$$+ 21(2bc_1d_1 + 2b^2e_1 + 3bc_2d_0 + 4c_0c_1c_2 - 4c_1^3 - 5bc_0d_2)x$$
$$+ \ldots,$$

$$\frac{d^2H}{dz^2} = 72b^2c_0 + 168b^2d_0x + \ldots$$

Whence $$\Phi = 6(b^4e_0 + 2b^2c_0c_1^2 - 2b^2c_0^2c_2 + 4b^3c_0d_1 - 4b^3c_1d_0)$$
$$+ \begin{Bmatrix} 6b^4f_0 + 24b^2c_0c_2d_0 + 48b^3c_1e_0 - 48b^3c_0e_1 \\ - 228b^2c_0^2d_2 - 252b^2c_1^2d_0 + 456b^2c_0c_1d_1 \end{Bmatrix} x$$
$$+ \ldots$$

We will now show that the equation of the bitangential is

$$[297HJ(u, H, \Phi) - 8208HJ'(u, H, \Phi) - 20J(u, H, \theta)]^2 = 75625(4\theta - 9H\Phi)^3,$$

where by $J(u, H, \varphi)$ we mean the Jacobian of these three functions; and $J'(u, H, \varphi)$ means that, in forming the Jacobian, φ is to be differentiated on the supposition that the second differential coefficients of H, which enter into the expression for φ, are constant.

We verify this result by showing that the absolute term vanishes.

We find without difficulty the absolute term in

$$4\theta - 9H\Phi = 36b^6d_0^2 - 54b^6c_0e_0 = 18b^6(2d_0^2 - 3c_0e_0).$$

To calculate the absolute term in J it is useful to observe that, if F is any function of x, y, z, the absolute term in $J(u, H, F)$ is

$$\begin{vmatrix} \frac{du}{dx}, & \frac{du}{dy}, & \frac{du}{dz} \\ \frac{dH}{dx}, & \frac{dH}{dy}, & \frac{dH}{dz} \\ \frac{dF}{dx}, & \frac{dF}{dy}, & \frac{dF}{dz} \end{vmatrix} = \begin{vmatrix} 0, & 5b, & 0 \\ 3b^2d_0, & \frac{dH}{dy}, & 9b^2c_0 \\ \frac{dF}{ax}, & \frac{dF}{dy}, & \frac{dF}{dz} \end{vmatrix}$$

$$= 5b\left\{9b^2c_0\frac{dF}{dx} - 3b^2d_0\frac{dF}{dz}\right\};$$

and similarly for J'.

Whence we have after reduction

$$J(u,\ H,\ \varphi) = 5b\left\{\begin{array}{l} 54b^6c_0f_0 - 234b^6d_0e_0 + 432b^5c_0c_1e_0 \\ - 432b^5c_0^2e_1 + 684b^4c_0^2c_2d_0 + 864b^5c_0c_1e_1 \\ + 936b^5c_1d_0^2 - 936b^5c_0d_0d_1 - 2052b^4c_0^3d_2 \\ - 2736b^4c_0c_1^2d_0 + 4104b^4c_0^2c_1d_1 \end{array}\right\} + \cdots,$$

$$J(u,\ H,\ \theta) = 5b\left\{\begin{array}{l} 324b^8c_0d_0e_0 - 594b^8d_0^3 + 972b_0^7c^3e_1 \\ - 972b^7c_0^2c_1e_0 + 1539b^6c_0^3c_2d_0 + 4050b^7c_0c_1d_0^2 \\ - 4050b^7c_0^2d_0d_1 - 4617b^6c_0^4d_2 - 6156b^6c_0^2c_1^2d_0 \\ + 9234b^6c_0^3c_1d_1 \end{array}\right\} + \cdots,$$

$$J'(u,\ H,\ \varphi) = 5b\left\{\begin{array}{l} - 6b^6d_0e_0 + 18b^5c_0c_1e_0 - 18b^5c_0^2e_1 \\ + 21b^4c_0^2c_2d_0 + 24b^5c_1d_0^2 - 24b^5c_0d_0d_1 \\ - 63b^4c_0^3d_2 - 84b^4c_0c_1^2d_0 + 126b^4c_0^2c_1d_1 \end{array}\right\} + \cdots.$$

Whence we find the absolute term in

$$297HJ(u,\ H,\ \varphi) - 8208HJ'(u,\ H,\ \varphi) - 20J(u,\ H,\ \theta)$$

to be $$2970b^9(27c_0^2f_0 - 45c_0d_0e_0 + 20d_0^3).$$

Therefore the absolute term in

$$[297HJ(u,\ H,\ \Phi) - 8208HJ'(u,\ H,\ \Phi) - 20J(u,\ H,\ \theta)]^2 - 75625(4\theta - 9H\Phi)^3 = 0.$$

is proportional to

$$b^{18}c_0^2(27c_0^2f_0^2 + 40d_0^3f_0 + 50c_0e_0^3 - 25d_0^2e_0^2 - 90c_0d_0e_0f_0),$$

and therefore vanishes, if $y = 0$ is a double tangent. That is, this curve passes through the points of contact of the double tangents.

It should be observed that the equation we have found for the bitangential contains an irrelevant factor of the eighteenth order in the variables and of the sixth in the coefficients of the given curve. This factor appears to be the square of the Hessian, and dividing out, the resulting equation is of the forty-eighth order in the variables and of the eighteenth in the coefficients of the original equation.

ON THE MOTION OF HYPERION.

By PROF. ORMOND STONE, University of Virginia, Va.

1. If we assume that the planes of the orbits of Titan and Hyperion coincide, the differential equations for the motion of the latter may be written

$$r^2 \frac{dw}{dt} = C + m'k^2 \int Srdt, \tag{1}$$

$$\frac{d^2r}{dt^2} - r\left(\frac{dw}{dt}\right)^2 + \frac{k^2}{r^2} = m'k^2R; \tag{2}$$

in which r and w are the radius vector and longitude in orbit of Hyperion, k the constant of the Saturnian system as derived from the orbit of Titan, m' the mass of Titan, $m'k^2R$ and $m'k^2S$ the components, in the plane of the orbit, of the disturbing force in the direction of and perpendicular to the radius vector, and C is a constant of integration.

We shall assume that r, w, R, and S can be developed into series of sines and cosines of multiples of $\theta = l' - l - \lambda$, having coefficients which are periodic functions of $\eta = 4l - 3l'$, in which l and l' are the mean longitudes in orbit of Hyperion and Titan, and λ is a constant to be derived from observation. η is evidently the longitude of the mean conjunction point of the two satellites, and $l' - l$ their mean elongation.

If we put

$$n = \frac{k}{a^{\frac{3}{2}}},$$

and

$$k\sqrt{a(1-\nu)} = C + m'k^2\int (Sr)_0 dt;$$

in which a, the mean distance of Hyperion from Saturn, is a constant, and $(Sr)_0$ is that portion of Sr which is independent of θ. ν is thus a function of η only.

We may also put

$$d\eta = (4n - 3n')dt,$$

and

$$C = k\sqrt{a(1-\nu_0)};$$

whence, if a', the semimajor axis of Titan's orbit, be taken as the unit of distance, we may write

$$\sqrt{(1-\nu)} = \sqrt{(1-\nu_0)} + \frac{m'}{\sqrt{(1+m')}}\frac{n'}{4n-3n'}\frac{1}{\sqrt{a}}\int (Sr)_0 d\eta,$$

which may be solved by a simple integration.

We may now write

$$r^2\frac{dw}{dt} = k\sqrt{a(1-\nu)} + \frac{m'}{1+m'}\,\frac{n'^2}{n'-n}\,U, \tag{3}$$

in which U is that portion of $\int Srd\theta$ which remains after omitting those terms which are independent of θ. As is evident,

$$d\theta = (n'-n)\,dt.$$

Substituting in (2) the value of dw/dt derived from (3), the former equation becomes

$$\frac{d^2r}{dt^2} - \frac{k^2a(1-\nu)}{r^3} + \frac{k^2}{r^2} = m'\frac{k^2}{a^2}P, \tag{4}$$

where, neglecting terms in m'^2,

$$P = a^2R + \frac{2\sqrt{(1-\nu)}}{s^3\sqrt{a}}\,\frac{n}{n'-n}\,U, \tag{5}$$

in which

$$s = \frac{r}{a}.$$

If we multiply by a^2/k^2, (4) becomes

$$\mu\frac{d^2s}{d\theta^2} + \frac{s-1+\nu}{s^3} = m'P, \tag{6}$$

in which

$$\mu = \frac{(n'-n)^2}{n^2}.$$

2. As the equations of motion of an intermediate orbit, we shall assume

$$r_{,} = a_{,}s_{,},$$

$$3(n'-n) = \frac{k^2}{a_{,}^3},$$

$$s_{,}^2\frac{dw_{,}}{3d\theta} = \sqrt{(1-e^2)}, \tag{7}$$

$$\tfrac{1}{9}\frac{d^2s_{,}}{d\theta^2} + \frac{s_{,}-1+e^2}{s_{,}^3} = 0; \tag{8}$$

i. e. a Keplerian ellipse with mean motion $3(n'-n)$, semimajor axis $a_{,}$, and eccentricity e; $a_{,}$ is a constant, and e a function of η only, satisfying the condition that the coefficient of $\cos 3\theta$ in δs is zero.

Neglecting powers of e greater than the fourth, we may write

$$\begin{aligned} s_{,} = 1 &+ \tfrac{1}{2}e^2 \\ &+ e(1 - \tfrac{3}{8}e^2)\cos 3\theta \\ &- \tfrac{1}{2}e^2(1 - \tfrac{2}{3}e^2)\cos 6\theta \\ &+ \tfrac{3}{8}e^3 \cos 9\theta \\ &- \tfrac{1}{3}e^4 \cos 12\theta, \end{aligned} \tag{9}$$

and

$$\begin{aligned} \frac{(1 - e^2)}{s_{,}^2} = 1 &- 2e(1 - \tfrac{1}{8}e^2)\cos 3\theta \\ &+ \tfrac{5}{2}e^2(1 - \tfrac{11}{30}e^2)\cos 6\theta \\ &- \tfrac{13}{4}e^3 \cos 9\theta \\ &+ \tfrac{103}{24}e^4 \cos 12\theta. \end{aligned}$$

Subtracting (8) from (6), neglecting powers and products of $\delta s = s - s_{,}$ and $\delta\nu = \nu - e^2$, and putting

$$\beta = \frac{3(1 - e^2)}{s_{,}^4} - \frac{2}{s_{,}^3},$$

we have

$$\begin{aligned} \mu\frac{d^2\delta s}{d\theta^2} + \delta\mu\frac{d^2 s_{,}}{d\theta^2} + \beta\delta s + \frac{\delta\nu}{s_{,}^3} &= m'P \\ &= m'\sum_0^\infty{}_i(P_i \cos i\theta + Q_i \sin i\theta), \end{aligned} \tag{10}$$

in which $\delta\mu = \mu - \frac{1}{9}$, $\delta\nu$ satisfies the condition that δs shall contain no term independent of θ, i is any positive whole number, and P_i and Q_i are assumed to be functions of η only.

3. Considering, for the present, coefficients of cos 0θ, cos 3θ, cos 6θ, and cos 9θ only, we may write

$$\begin{aligned} \beta = 1 &+ 3e^2 \\ &- 6e(1 + \tfrac{9}{8}e^2)\cos 3\theta \\ &+ 12e^2 \cos 6\theta \\ &- \tfrac{85}{4}e^3 \cos 9\theta. \end{aligned}$$

If we assume

$$\delta s = \sum_1^\infty{}_i(a_i \cos i\theta + g_i \sin i\theta), \tag{11}$$

and substitute for $s_{,}$ and δs their values given by (9) and (11), remembering that $a_3 = 0$, we have

$$
\begin{aligned}
\frac{d^2\delta s}{d\theta^2} &= -36a_6 \cos 6\theta - 81a_9 \cos 9\theta, \\
\frac{d^2 s_{,}}{d\theta^2} &= -9e(1 - \tfrac{3}{8}e^2) \cos 3\theta + 18e^2 \cos 6\theta, \\
\beta\delta s &= \quad 6e^2 a_6 \\
&\quad - [3e(1 + \tfrac{14}{3}e^2)a_6 - 6e^2 a_9] \cos 3\theta \\
&\quad + [(1 + 3e^2)a_6 - 3ea_9] \cos 6\theta \\
&\quad - (3ea_6 - a_9) \cos 9\theta, \\
\frac{1}{s_{,}^3} &= \quad 1 + \tfrac{3}{2}e^2 \\
&\quad - 3e(1 + \tfrac{9}{8}e^2) \cos 3\theta \\
&\quad + \tfrac{9}{2}e^2 \cos 6\theta.
\end{aligned}
$$

Substituting in (10) and separating by indeterminate coefficients, we obtain

$$
\begin{aligned}
6e^2 a_6 + (1 + \tfrac{3}{2}e^2)\delta\nu &= m'P_0, \\
-9e(1 - \tfrac{3}{8}e^2)\delta\mu - 3e(1 + \tfrac{14}{3}e^2)a_6 + 6e^2 a_9 - 3e(1 + \tfrac{9}{8}e^2)\delta\nu &= m'P_3, \\
18e^2\delta\mu - 3(1 - e^2 + 12\delta\mu)a_6 - 3ea_9 + \tfrac{9}{2}e^2\delta\nu &= m'P_6, \\
-3ea_6 - 8a_9 &= m'P_9;
\end{aligned}
$$

or

$$
\begin{aligned}
\delta\nu &= (1 - \tfrac{3}{2}e^2)m'P_0 - 6e^2 a_6, \\
9e\delta\mu &= -3e(1 + \tfrac{3}{2}e^2)\delta\nu - (1 + \tfrac{3}{8}e^2)m'P_3 - 3e(1 + \tfrac{121}{24}e^2)a_6 + 6e^2 a_9, \\
a_6 &= \tfrac{3}{2}e^2\delta\nu + 6e^2\delta\mu - \tfrac{1}{3}(1 + e^2 - 12\delta\mu)m'P_6 - ea_9, \\
a_9 &= -\tfrac{3}{8}ea_6 - \tfrac{1}{8}m'P_9;
\end{aligned}
$$

whence, solving by trial,

$$
\begin{aligned}
m'^{-1}\delta\nu &= (1 - \tfrac{3}{2}e^2)P_0 + 2e^2 P_6, \\
9m'^{-1}\delta\mu . e &= -3e(1 - \tfrac{1}{2}e^2)P_0 - (1 - \tfrac{13}{8}e^2)P_3 + e(1 - \tfrac{5}{6}e^2 + 12\delta\mu)P_6 - \tfrac{9}{8}e^2 P_9, \\
m'^{-1}a_6 &= -\tfrac{1}{2}e^2 P_0 - \tfrac{2}{3}eP_3 - \tfrac{1}{3}(1 - \tfrac{5}{8}e^2 - 12\delta\mu)P_6 + \tfrac{1}{8}eP_9, \qquad (12) \\
m'^{-1}a_9 &= \tfrac{1}{8}eP_6 - \tfrac{1}{8}P_9.
\end{aligned}
$$

For the succeeding terms, we may write

$$a_{3i} = \frac{1}{1 - i^2} m' P_{3i}. \tag{13}$$

4. Considering cosine terms which involve only those multiples of θ which are not also multiples of 3θ, equation (10) may be written, with sufficient accuracy,

$$\tfrac{1}{9} \frac{d^2 \delta s}{d\theta} + \beta \delta s = m' \Sigma_i P_i \cos i\theta,$$

in which we may assume

$$\delta s = a_1 \cos \theta + a_2 \cos 2\theta + a_4 \cos 4\theta + \ldots$$

and

$$\beta = 1 + 2\beta_3 \cos 3\theta + 2\beta_6 \cos 6\theta;$$

whence, separating by indeterminate coefficients, we have

$$\left.\begin{array}{l}
\frac{8}{9} a_1 + \beta_3 a_2 + \beta_3 a_4 + \beta_6 a_5 + \beta_6 a_7 = m' P_1, \\
\beta_3 a_1 + \frac{5}{9} a_2 + \beta_6 a_4 + \beta_3 a_5 + \beta_6 a_8 = m' P_2, \\
\beta_3 a_1 + \beta_6 a_2 - \frac{7}{9} a_4 + \beta_3 a_7 + \beta_6 a_{10} = m' P_4, \\
\beta_6 a_1 + \beta_3 a_2 - \frac{16}{9} a_5 + \beta_3 a_8 + \beta_6 a_{11} = m' P_5, \\
\beta_6 a_1 + \beta_3 a_4 - \frac{40}{9} a_7 + \beta_3 a_{10} + \beta_6 a_{13} = m' P_7, \\
\beta_6 a_2 + \beta_3 a_5 - \frac{55}{9} a_8 + \beta_3 a_{11} + \beta_6 a_{14} = m' P_8, \\
\ldots \ldots \ldots \ldots \ldots \ldots
\end{array}\right\} \tag{14}$$

Putting $p_i = m' P_i$, and calling Δ the determinant whose constituents are the coefficients of a_1, a_2, etc.; Δ_1 the determinant formed by substituting p_1, p_2, etc. for the first column in Δ, i. e. for the coefficients of a_1; Δ_2 the determinant formed by substituting p_1, p_2, etc. for the coefficients of a_2, etc., we have

$$a_1 = \frac{\Delta_1}{\Delta}; \quad a_2 = \frac{\Delta_2}{\Delta}; \quad a_4 = \frac{\Delta_4}{\Delta}; \quad \text{etc.}$$

To expand these determinants, let us consider the determinant in its most general form. If we put

$$A_{ijk\ldots} = \frac{1}{a_{ii}\, a_{jj}\, a_{kk} \ldots} \begin{vmatrix} 0 & a_{ji} & a_{ki} & \ldots \\ a_{ij} & 0 & a_{kj} & \ldots \\ a_{ik} & a_{jk} & 0 & \ldots \\ \ldots & \ldots & \ldots & \ldots \end{vmatrix},$$

in which $i < j < k < \ldots$, we have

$$\begin{vmatrix} a_{11} & a_{21} & a_{31} & \ldots \\ a_{12} & a_{22} & a_{32} & \ldots \\ a_{13} & a_{23} & a_{33} & \ldots \\ \ldots & \ldots & \ldots & \ldots \end{vmatrix} = a_{11}\, a_{22}\, a_{33} \ldots (1 + \Sigma A_{ij} + \Sigma A_{ijk} + \Sigma A_{ijkl} + \ldots), \quad (15)$$

as may readily be seen by transforming and generalizing the method for expanding in products of the leading constituents given in Burnside and Panton's Theory of Equations, p. 246.

In the expansion of Δ we find

$$\Sigma A_{ij} = -A_2\beta_3^2 + \ldots,$$

in which

$$\begin{aligned} A_2 = {} & \tfrac{9}{8}\cdot\tfrac{9}{5} - \tfrac{9}{5}\cdot\tfrac{9}{16} + \tfrac{9}{16}\cdot\tfrac{9}{55} + \ldots \\ & - \tfrac{9}{8}\cdot\tfrac{9}{7} + \tfrac{9}{7}\cdot\tfrac{9}{40} + \tfrac{9}{40}\cdot\tfrac{9}{91} + \ldots \\ = {} & \sum_0^\infty{}_i \left(\frac{1}{1-(i-\frac{1}{3})^2}\cdot\frac{1}{1-(i+\frac{2}{3})^2} + \frac{1}{1-(i+\frac{1}{3})^2}\cdot\frac{1}{1-(i+\frac{4}{3})^2} \right) \\ = {} & \tfrac{1}{6} \sum_0^\infty{}_i \left(\begin{matrix} \dfrac{3}{1+(i-\frac{1}{3})} - \dfrac{1}{1-(i-\frac{1}{3})} - \dfrac{1}{1+(i+\frac{2}{3})} + \dfrac{3}{1-(i+\frac{2}{3})} \\ + \dfrac{3}{1+(i+\frac{1}{3})} - \dfrac{1}{1-(i+\frac{1}{3})} - \dfrac{1}{1+(i+\frac{4}{3})} + \dfrac{3}{1-(i+\frac{4}{3})} \end{matrix} \right) \\ = {} & 0; \end{aligned}$$

whence, with sufficient accuracy,

$$\begin{aligned} \Delta &= \tfrac{8}{9}\cdot\tfrac{5}{9}\cdot - \tfrac{7}{9}\cdot - \tfrac{16}{9} \ldots \\ &= \tfrac{8}{9} \prod_1^\infty{}_i \left[1-(i-\tfrac{1}{3})^2\right]\left[1-(i+\tfrac{1}{3})^2\right]. \end{aligned}$$

Expanding Δ_1, we obtain

$$\Delta_1 = p_1\cdot\tfrac{5}{9}\cdot - \tfrac{7}{9}\cdot - \tfrac{16}{9} \ldots \left(\begin{matrix} 1 + \frac{9}{8}\,(\frac{9}{5} - \frac{9}{7})\,\beta_3^2 \\ - \frac{9}{5}\,\beta_3 \dfrac{p_2}{p_1} + \frac{9}{7}\,\beta_3 \dfrac{p_4}{p_1} \\ + \frac{9}{16}\,\beta_6 \dfrac{p_5}{p_1} + \frac{9}{40}\,\beta_6 \dfrac{p_7}{p_1} \end{matrix} \right);$$

whence

$$
\begin{aligned}
m'^{-1}a_1 = \quad & \tfrac{9}{8}\left[1 + \tfrac{9}{8}\left(\tfrac{9}{5} - \tfrac{9}{7}\right)\beta_3^2\right]P_1 \\
& - \tfrac{9}{8}\cdot\tfrac{9}{5}\,\beta_3 P_2 + \tfrac{9}{8}\cdot\tfrac{9}{7}\,\beta_3 P_4 \\
& + \tfrac{9}{8}\cdot\tfrac{9}{16}\,\beta_6 P_5 + \tfrac{9}{8}\cdot\tfrac{9}{40}\,\beta_6 P_7,
\end{aligned}
$$

in which the coefficient of β_3^2 is equal to A_2 after the omission of those terms in A_2 which contain as factors the reciprocals of such constituents as are situated at the intersection of the p-column and the diagonal; in other words, since $A_2 = 0$, it is equal to the sum of the terms mentioned taken with opposite signs.

In general, when i is not a multiple of 3, we have

$$
\begin{aligned}
m'^{-1}a_i = \quad & \frac{9}{9-i^2}\left[1 + \frac{9}{9-i^2}\left\{\frac{9}{9-(i-3)^2} + \frac{9}{9-(i+3)^2}\right\}\beta_3^2\right]P_i \\
& - \frac{9}{9-i^2}\cdot\frac{9}{9-(i-3)^2}\,\beta_3 P_{i-3} - \frac{9}{9-i^2}\cdot\frac{9}{9-(i+3)^2}\,\beta_3 P_{i+3} \qquad (16) \\
& - \frac{9}{9-i^2}\cdot\frac{9}{9-(i-6)^2}\,\beta_6 P_{i-6} - \frac{9}{9-i^2}\cdot\frac{9}{9-(i+6)^2}\,\beta_6 P_{i+6},
\end{aligned}
$$

in which

$$
\begin{aligned}
P_{-j} &= P_{+j}, \\
\beta_3 &= -3e\left(1 + \tfrac{9}{8}e^2\right), \\
\beta_6 &= 6e^2.
\end{aligned}
$$

5. For the terms involving only sines of multiples of 3θ, equation (10) may be written

$$
\mu\frac{d^2\delta s}{d\theta^2} + \beta\delta s = m'\sum_1^\infty{}_i Q_{3i}\sin 3i\theta,
$$

in which we may assume

$$
\begin{aligned}
\delta s &= \sum_1^\infty{}_i\, g_{3i}\sin 3i\theta, \\
\beta &= \beta_0 + 2\sum_1^\infty{}_i \beta_{3i}\cos 3i\theta, \\
m'Q_{3i} &= q_{3i};
\end{aligned}
$$

whence, separating by indeterminate coefficients, we have

$$
\begin{aligned}
\lambda_3 g_3 + (\beta_3 - \beta_9)g_6 + (\beta_6 - \beta_{12})g_9 + (\beta_9 - \beta_{15})g_{12} + \ldots &= q_3, \\
(\beta_3 - \beta_9)g_3 + \qquad \lambda_6 g_6 + (\beta_3 - \beta_{15})g_9 + (\beta_6 - \beta_{18})g_{12} + \ldots &= q_6,
\end{aligned}
$$

$$(\beta_6 - \beta_{12})g_3 + (\beta_3 - \beta_{15})g_6 + \qquad \lambda_9 g_9 + (\beta_3 - \beta_{21})g_{12} + \ldots = q_9,$$

$$(\beta_9 - \beta_{15})g_3 + (\beta_6 - \beta_{18})g_6 + (\beta_3 - \beta_{21})g_9 + \qquad \lambda_{12} g_{12} + \ldots = q_{12},$$

. . . . ;

where, if $\sigma = 9\delta\mu$,

$$\lambda_3 = -3e^2 - \sigma + \tfrac{11}{8}e^4,$$
$$\lambda_6 = -3 + 3e^2 - 4\sigma,$$
$$\lambda_9 = -8,$$
$$\lambda_{12} = -15.$$

If D be the determinant whose constituents are the coefficients of g_3, g_6, etc.; D_3 the determinant found by substituting q_3, q_6, etc. for the first column in D, i. e. for the coefficients of g_3, etc., we have, as in the preceding section,

$$g_3 = \frac{D_3}{D}; \qquad g_6 = \frac{D_6}{D}; \qquad \text{etc.}$$

Applying formula (15), we obtain

$$D = \lambda_3\lambda_6\lambda_9\ldots\left(1 - \frac{(\beta_3 - \beta_9)^2}{\lambda_3\lambda_6} - \frac{\beta_6^2}{\lambda_3\lambda_9} - \frac{\beta_3^2}{\lambda_6\lambda_9} - \frac{\beta_3^2}{\lambda_9\lambda_{12}} + \frac{2\beta_3^2\beta_6}{\lambda_3\lambda_6\lambda_9} + \frac{\beta_3^4}{\lambda_3\lambda_6\lambda_9\lambda_{12}}\right),$$

$$D_3 = q_3\lambda_6\lambda_9\ldots\left(1 - \frac{\beta_3 q_6}{\lambda_6 q_3} - \frac{\beta_6 q_9}{\lambda_9 q_3} - \frac{\beta_3^2}{\lambda_6\lambda_9} - \frac{\beta_3^2}{\lambda_9\lambda_{12}} + \frac{2\beta_3^2 q_9}{\lambda_6\lambda_9 q_3}\right),$$

$$D_6 = \lambda_3 q_6\lambda_9\ldots\left(1 - \frac{\beta_3 q_3}{\lambda_3 q_6}\right),$$

$$D_9 = \lambda_3\lambda_6 q_9\lambda_{12}\ldots\left(1 - \frac{\beta_3^2}{\lambda_3\lambda_6} - \frac{\beta_6 q_3}{\lambda_3 q_9} + \frac{2\beta_3^2 q_3}{\lambda_3\lambda_6 q_9}\right),$$

$$D_{12} = \lambda_3\lambda_6\lambda_9 q_{12}\lambda_{15}\ldots\left(1 - \frac{\beta_3^2}{\lambda_3\lambda_6}\right),$$

. . . . ;

whence, substituting for λ_3, β_3, etc. within the parentheses their values in terms of σ and e,

$$D = -\sigma\lambda_6\lambda_9\ldots(1 + \tfrac{71}{20}e^2),$$
$$D_3 = \lambda_6\lambda_9\ldots[(1 - \tfrac{9}{20}e^2)q_3 - eq_6 + \tfrac{3}{2}q_9],$$
$$D_6 = \lambda_9\lambda_{12}\ldots[3eq_3 - (3e^2 + \sigma)q_6],$$
$$D_9 = \lambda_6\lambda_{12}\lambda_{15}\ldots(-12e^2 q_3 - \sigma q_9),$$
$$D_{12} = \lambda_6\lambda_9\lambda_{15}\lambda_{18}\ldots(-\sigma q_{12}),$$

. . . .

Substituting $m'Q_{3i}$ for q_{3i} and dividing by $m'D/\sigma$, we have

$$
\begin{aligned}
m'^{-1}\sigma g_3 &= -(1-4e^2)Q_3 + eQ_6 - \tfrac{3}{2}e^2Q_9, \\
m'^{-1}\sigma g_6 &= \quad eQ_3 - (e^2 + \tfrac{1}{3}\sigma)Q_6, \\
m'^{-1}\sigma g_9 &= -\tfrac{3}{2}e^2Q_3 - \tfrac{1}{8}\sigma Q_9;
\end{aligned}
\tag{17}
$$

and, for succeeding terms,

$$g_{3i} = \lambda_{3i}^{-1}Q_{3i} = \frac{1}{1-i^2}\, m'Q_{3i}. \tag{18}$$

6. For the sine terms not involving multiples of 3θ, we may assume $\mu = \frac{1}{9}$ and $\beta = 1$; whence, substituting in (10) and separating by intederminate coefficients,

$$g_i = \frac{9}{9-i^2}\, m'Q_i. \tag{19}$$

7. Instead of (3), we may write

$$
\begin{aligned}
s^2\frac{dw}{\mathrm{n}dt} &= \sqrt{(1-\nu)} + \frac{m'}{\sqrt{a}}\,\frac{n'}{n'-n}\,U \\
&= \sqrt{(1-\nu)} + m'\sum_1^\infty{}_i(S_i\cos i\theta + T_i\sin i\theta),
\end{aligned}
\tag{20}
$$

in which S_i and T_i are assumed to be functions of η only. We have, also,

$$s_\prime^2\frac{dw_\prime}{3d\theta} = \sqrt{(1-e^2)}. \tag{7}$$

If we subtract (7) from (20), neglecting powers and products of δs, $\delta\nu = \nu - e^2$, and

$$\delta\frac{dw}{\mathrm{n}dt} = \frac{dw}{\mathrm{n}dt} - \frac{dw_\prime}{3d\theta},$$

we obtain

$$2s_\prime\frac{dw_\prime}{3d\theta}\,\delta s + s_\prime^2\delta\frac{dw}{\mathrm{n}dt} = -\tfrac{1}{2}(1+\tfrac{1}{2}e^2)\,\delta\nu + m'\sum_1^\infty{}_i(S_i\cos i\theta + T_i\sin i\theta), \tag{21}$$

in which

$$
\begin{aligned}
s_\prime\frac{dw_\prime}{3d\theta} &= 1 - \tfrac{1}{2}e^2 - e\cos 3\theta + e^2\cos 6\theta, \\
s_\prime^2 &= 1 + \tfrac{3}{2}e^2 + 2e\cos 3\theta - \tfrac{1}{2}e^2\cos 6\theta.
\end{aligned}
$$

Assuming

$$\delta\frac{dw}{\mathrm{n}dt} = \sum_0^\infty{}_i(n_i\cos i\theta + h_i\sin i\theta), \tag{22}$$

in which n_i and h_i are functions of η only, substituting in (21), and separating by indeterminate coefficients, we have, approximately,

$$\begin{aligned}
e^2a_6 + (1+\tfrac{3}{2}e^2)n_0 + en_3 - \tfrac{1}{4}e^2n_6 &= -\tfrac{1}{2}(1+\tfrac{1}{2}e^2)\delta\nu,\\
-ea_6 + \quad 2en_0 + n_3 + \quad en_6 &= m'S_3,\\
2a_6 \qquad\qquad + \quad n_6 &= m'S_6;
\end{aligned}$$

the solution of which gives

$$\begin{aligned}
n_0 &= -\tfrac{1}{2}(1+e^2)\delta\nu - \tfrac{9}{2}e^2a_6 - em'S_3 + \tfrac{5}{4}e^2m'S_6,\\
n_3 &= \qquad e\delta\nu + 3ea_6 + m'S_3 - \quad em'S_6,\\
n_6 &= \qquad\qquad - 2a_6 + \qquad m'S_6;
\end{aligned}\tag{23}$$

or, substituting from equations (12),

$$\begin{aligned}
m'^{-1}n_0 &= -\tfrac{1}{2}(1-\tfrac{1}{2}e^2)P_0 + \tfrac{1}{2}e^2P_6 - eS_3 + \tfrac{5}{4}e^2S_6,\\
m'^{-1}n_3 &= \qquad eP_0 - \quad eP_6 + S_3 - \quad eS_6,\\
m'^{-1}n_6 &= \qquad\qquad \tfrac{2}{3}P_6 \qquad + \quad S_6.
\end{aligned}\tag{23'}$$

Also,

$$\begin{aligned}
2(1-e^2)g_3 - eg_6 + (1+\tfrac{7}{4}e^2)h_3 + eh_6 \qquad &= m'T_3,\\
-eg_3 + 2g_6 + \qquad eh_3 + h_6 \qquad &= m'T_6,\\
2g_9 - \qquad \tfrac{1}{4}e^2h_3 + eh_6 + h_9 &= m'T_9;
\end{aligned}\tag{24}$$

whence

$$\begin{aligned}
h_3 &= m'T_3 - 2(1-\tfrac{5}{4}e^2)g_3 + 3eg_6,\\
h_6 &= m'T_6 + \qquad 3eg_3 - \quad 2g_6,\\
h_9 &= m'T_9 - \qquad \tfrac{7}{2}e^2g_3 + 2eg_6 - 2g_9;
\end{aligned}\tag{24'}$$

or, substituting from equations (17),

$$\begin{aligned}
m'^{-1}\sigma h_3 &= \sigma T_3 + 2(1-\tfrac{15}{4}e^2)Q_3 - 2eQ_6 + 3e^2Q_9,\\
m'^{-1}\sigma h_6 &= \sigma T_6 - \qquad 5eQ_3 + 3e^2Q_6,\\
m'^{-1}\sigma h_9 &= \sigma T_9 + \qquad \tfrac{17}{2}e^2Q_3.
\end{aligned}$$

For the remaining terms, we may write, in general,

$$2a_i - ea_{i+3} - ea_{i-3} + n_i + en_{i+3} + en_{i-3} = m'S_i, \tag{25}$$

$$2g_i - eg_{i+3} - eg_{i-3} + h_i + eh_{i+3} + eh_{i-3} = m'T_i, \tag{26}$$

in which

$$a_{i-3} = a_{-i+3}, \qquad n_{i-3} = n_{-i+3},$$

$$g_{i-3} = -g_{-i+3}, \qquad h_{i-3} = -h_{-i+3};$$

whence, approximately,

$$n_{i\pm 3} = m'S_{i\pm 3} - 2a_{i\pm 3},$$

$$h_{i\pm 3} = m'T_{i\pm 3} - 2g_{i\pm 3};$$

which, substituted in (25) and (26), give

$$n_i = m'S_i - em'S_{i+3} - em'S_{i-3} - 2a_i + 3ea_{i+3} + 3ea_{i-3}, \tag{27}$$

$$h_i = m'T_i - em'T_{i+3} - em'T_{i-3} - 2g_i + 3eg_{i+3} + 3eg_{i-3}, \tag{28}$$

in which

$$S_{i-3} = S_{-i+3}, \qquad T_{i-3} = -T_{-i+3}.$$

8. If we omit periodic terms, we may write

$$\left(\frac{dw}{\mathrm{n}dt}\right)_0 = \left(\frac{dw,}{3d\theta}\right)_0 + \left(\delta\frac{dw}{\mathrm{n}dt}\right)_0,$$

or

$$\frac{n}{\mathrm{n}} = 1 + Am', \tag{29}$$

in which A is the constant part of $m'^{-1}n_0$, which is given by the first of equations (23′). We may also put

$$\sigma = 9\delta\mu = 9\,\frac{(n' - n)^2}{\mathrm{n}^2} - 1 = Bm', \tag{30}$$

in which B is obtained from the second of equations (12).

Putting

$$\gamma = \frac{3(n' - n)}{n} - 1 = \frac{3n' - 4n}{n}, \tag{31}$$

and substituting in (30), we have

$$(1 + \gamma)^2(1 + Am')^2 = 1 + Bm';$$

or, with sufficient accuracy,

$$2\gamma + \gamma^2 + 2Am' = Bm';$$

whence

$$m' = \frac{2\gamma + \gamma^2}{B - 2A}. \tag{32}$$

9. We may now put

$$\frac{dw}{\mathrm{n}dt} = \frac{n}{\mathrm{n}} + n_\eta + \left(\frac{dw}{\mathrm{n}dt}\right)_\theta$$

$$= \frac{n}{\mathrm{n}} + n_\eta + \left(\frac{dw_\prime}{3d\theta}\right)_\theta + \left(\delta\frac{dw}{\mathrm{n}dt}\right)_\theta, \qquad (33)$$

in which

$$n_\eta = n_0 - Am'.$$

The subscript θ indicates that only terms involving that quantity are considered.

Multiplying by $\mathrm{n}dt$ and integrating,

$$w = w_0 + nt + \frac{\mathrm{n}}{4n - 3n'}\int n_\eta d\eta + \frac{\mathrm{n}}{3(n'-n)}(w_\prime)_\theta + \frac{\mathrm{n}}{n'-n}\int\left(\delta\frac{dw}{\mathrm{n}dt}\right)_\theta d\theta; \quad (34)$$

or, with sufficient accuracy,

$$w = w_0 + nt - \frac{1}{\gamma}\int n_\eta d\eta + (1 - Am' - \gamma)(w_\prime)_\theta + 3\int\left(\delta\frac{dw}{\mathrm{n}dt}\right)_\theta d\theta. \quad (34')$$

Since

$$d\eta = \mathrm{g}d\theta,$$

where

$$\mathrm{g} = \frac{4n - 3n'}{n' - n} = -\frac{3\gamma}{1+\gamma},$$

we have, by integration,

$$\eta = \mathrm{g}\theta + \mathrm{g}_0;$$

whence

$$\cos j\eta \cos i\theta = \tfrac{1}{2}\cos(i\theta + j\eta) + \tfrac{1}{2}\cos(i\theta - j\eta)$$

$$= \tfrac{1}{2}\cos[(i + j\mathrm{g})\theta + j\mathrm{g}_0] + \tfrac{1}{2}\cos[(i - j\mathrm{g})\theta - j\mathrm{g}_0],$$

and

$$\int\cos j\eta \cos i\theta\, d\theta = \tfrac{1}{2}\frac{1}{i + j\mathrm{g}}\sin(i\theta + j\eta) + \tfrac{1}{2}\frac{1}{i - j\mathrm{g}}\sin(i\theta - j\eta)$$

$$= \frac{i}{i^2 - j^2\mathrm{g}}\cos j\eta \sin i\theta - \frac{j\mathrm{g}}{i^2 - j^2\mathrm{g}^2}\sin j\eta \cos i\theta. \qquad (35)$$

Similarly

$$\int\sin j\eta \cos i\theta\, d\theta = \frac{i}{i^2 - j^2\mathrm{g}^2}\sin j\eta \sin i\theta + \frac{j\mathrm{g}}{i^2 - j^2\mathrm{g}^2}\cos j\eta \cos i\theta, \quad (36)$$

$$\int\cos j\eta \sin i\theta\, d\theta = -\frac{i}{i^2 - j^2\mathrm{g}^2}\cos j\eta \cos i\theta - \frac{j\mathrm{g}}{i^2 - j^2\mathrm{g}^2}\sin j\eta \sin i\theta, \quad (37)$$

$$\int\sin j\eta \sin i\theta\, d\theta = -\frac{i}{i^2 - j^2\mathrm{g}^2}\sin j\eta \cos i\theta + \frac{j\mathrm{g}}{i^2 - j^2\mathrm{g}^2}\cos j\eta \sin i\theta. \quad (38)$$

Since, however, g is small and is multiplied by small coefficients, it may, in general, be neglected; whence (34′) becomes

$$w = w_0 + nt - \frac{1}{\gamma}\int n_\eta d\eta + (1 - Am' - \gamma)(w_\prime)_\theta + 3\sum_1^\infty{}_i \frac{1}{i}(n_i \sin i\theta - h_i \cos i\theta). \quad (34'')$$

Similarly, in integrating $Srd\theta$, the coefficients of sines and cosines of multiples of θ may, in general, be considered as constant.

10. Differentiating

$$\sqrt{(1-\nu)} = \sqrt{(1-\nu_0)} + \frac{m'}{\sqrt{a}} \frac{n'}{4n - 3n'} \int (Sr)_0 d\eta \quad (39)$$

with regard to η, we have

$$-\tfrac{1}{2}(1 + \tfrac{1}{2}\nu)\frac{d\nu}{d\eta} = -e(1 + \tfrac{1}{2}e^2)\frac{de}{d\eta} - \tfrac{1}{2}\frac{d\delta\nu}{d\eta} = \frac{m'}{\sqrt{a}}\frac{n'}{4n - 3n'}(Sr)_0,$$

or

$$(Sr)_0 = -\frac{\sqrt{a}}{m'}\frac{4n - 3n'}{n'}\left[e(1 + \tfrac{1}{2}e^2)\frac{de}{d\eta} + \tfrac{1}{2}\frac{d\delta\nu}{d\eta}\right], \quad (40)$$

and may assume

$$e = e_0(1 + e_1 \cos\eta + f_1 \sin\eta + e_2 \cos 2\eta + f_2 \sin 2\eta + \ldots), \quad (41)$$

in which e_0, e_1, f_1, etc. are constants.

11. The coefficients of sines and cosines of multiples of η in e, $\delta\nu$, a_i, n_i, g_i, and h_i are found by substituting (41) in equations (12), (17), and (23)–(26), and separating by indeterminate coefficients. With assumed values of e_0, P_0, P_3, etc., the numerical values of B and of the coefficients in the expression for e/e_0 are first obtained from the second of equations (12); the numerical values of the other coefficients are then found by substituting these in the remainder of the equations mentioned.

12. For the computation of the forces we have the well-known formulæ,

$$\rho^2 = r'^2 \sin^2(w' - w) + [r' \cos(w' - w) - r]^2,$$

$$h = \frac{1}{\rho^3} - \frac{1}{r'^3},$$

$$R = hr' \cos(w' - w) - \frac{r}{\rho^3},$$

$$S = hr' \sin(w' - w),$$

in which ρ is the distance between the two satellites, and r' and w' are the radius vector and longitude in orbit of Titan.

Assuming an approximate orbit, a^2R and Sr are computed for different values

of η and θ. In so doing η and θ may be treated as constants. The assumption, however, that η is constant is equivalent to saying that the conjunction point is stationary, or, in other words, that the mean motions of the two satellites are commensurable. If t_c be the mean time of conjunction of the two satellites, we have

$$\theta = l' - l - \lambda = \mathrm{l}' - \mathrm{l} - \lambda,$$

in which

$$\mathrm{l} = \eta + 3(n' - n)(t - t_c)$$

and

$$\mathrm{l}' = \eta + 4(n' - n)(t - t_c)$$

are the mean longitudes of Hyperion and Titan on the assumption that η is constant.

Having determined the values of a^2R and Sr for individual values of θ and η, the former are expressed in series of sines and cosines of the latter by means of mechanical quadratures. With the values of a^2R and Sr thus expressed P_i, Q_i, S_i, T_i, are obtained and substituted in equations (12), (17), and (23)–(26), whence are derived the inequalities in r and w. The problem is solved when we have found expressions for r and w which agree with the assumed values of those quantities. The value of $(Sr)_0$ found by mechanical quadratures must also agree with that obtained by means of equation (40).

13. Assuming the perisaturnium of Titan as the origin of longitudes; putting

$$\lambda = 0,$$

$$n = 16^\circ.9199, \qquad n' = 22^\circ.5770,$$

$$\nu = e^2 = 0.01, \qquad e' = 0.0284,$$

$$\log a = 0.08351, \qquad a' = 1;$$

and neglecting δs and $\delta \dfrac{dw}{\mathrm{n}dt}$, the following values of a^2R and Sr were obtained for $\eta = 0$ and $\eta = 180^\circ$:—

θ	$\eta = 0^\circ$		$\eta = 180^\circ$.	
	a^2R	Sr	a^2R	Sr
0.0°	− 12.64	0.00	− 17.20	0.00
7.5	− 8.17	+ 3.23	− 11.22	+ 4.71
15.0	− 4.32	+ 2.02	− 5.76	+ 3.42
22.5	− 2.83	+ 1.07	− 3.61	+ 1.87
30.0	− 2.15	+ 0.52	− 2.68	+ 0.91
37.5	− 1.81	+ 0.24	− 2.17	+ 0.35
45.0	− 1.63	+ 0.13	− 1.84	+ 0.02
52.5	− 1.56	+ 0.07	− 1.63	− 0.12
60.0	− 1.58	+ 0.01	− 1.42	− 0.25
67.5	− 1.52	− 0.11	− 1.28	− 0.33
75.0	− 1.40	− 0.33	− 1.14	− 0.44
82.5	− 1.18	− 0.58	− 0.98	− 0.54
90.0	− 0.86	− 0.85	− 0.78	− 0.72
97.5	− 0.46	− 1.09	− 0.57	− 0.81
105.0	− 0.12	− 1.09	− 0.28	− 1.00
112.5	+ 0.20	− 1.06	+ 0.03	− 1.08
120.0	+ 0.46	− 0.96	+ 0.34	− 1.06
127.5	+ 0.66	− 0.84	+ 0.67	− 0.96
135.0	+ 0.81	− 0.68	+ 0.93	− 0.77
142.5	+ 0.93	− 0.53	+ 1.09	− 0.53
150.0	+ 1.03	− 0.39	+ 1.15	− 0.31
157.5	+ 1.11	− 0.27	+ 1.14	− 0.15
165.0	+ 1.16	− 0.17	+ 1.11	− 0.06
172.5	+ 1.19	− 0.08	+ 1.07	− 0.02
180.0	+ 1.06	0.00	+ 1.21	0.00

For negative values of θ, the signs of Sr change, but not those of a^2R.

Considering terms involving multiples of 3θ only, the following values of a^2R and Sr give

$$a^2R = -1.326 - 1.224\cos 3\theta - 0.847\cos 6\theta - 0.597\cos 9\theta - 0.323\cos 12\theta$$

$$+(0.167 + 0.276\cos 3\theta + 0.158\cos 6\theta + 0.084\cos 9\theta + 0.043\cos 12\theta)\cos\eta_{\prime}$$

$$Sr = \quad + 0.633\sin 3\theta + 0.534\sin 6\theta + 0.356\sin 9\theta + 0.230\sin 12\theta$$

$$+(0.103\sin 3\theta - 0.091\sin 6\theta - 0.069\sin 9\theta - 0.039\sin 12\theta)\cos\eta_{\prime}.$$

We have, also,

$$\frac{\sqrt{(1-\nu)}}{3\sqrt{a}}\cdot\frac{n'}{n'-n} = 1.202$$

and

$$\frac{1}{s^3} = \frac{1}{s_{\prime}^3} = 1.015 - 0.303\cos 3\theta + 0.045\cos 6\theta - 0.007\cos 9\theta + 0.001\cos 12\theta;$$

whence

$$P_0 = -1.108 + 0.132\cos\eta_{\prime},$$

$$P_3 = -2.709 + 0.516\cos\eta_{\prime}, \qquad S_3 = -0.765 + 0.124\cos\eta_{\prime},$$

$$P_6 = -1.225 + 0.221\cos\eta_{\prime}, \qquad S_6 = -0.322 + 0.054\cos\eta_{\prime},$$

$$P_9 = -0.806 + 0.130\cos\eta_{\prime}, \qquad S_9 = -0.145 + 0.029\cos\eta_{\prime}.$$

Substituting in the first of equations (23′), we have

$$m'^{-1}n_0 = 0.639 - 0.069\cos\eta_{\prime};$$

whence

$$A = 0.639 \qquad \text{and} \qquad m'^{-1}n_{\eta} = -0.069\cos\eta_{\prime}.$$

The assumed values of n and n' give $12\delta\mu = 0.0081$; whence, by the second of equations (12),

$$9m'^{-1}\delta\mu\,.\,e = 2.883 - 0.526\cos\eta_{\prime};$$

whence, assuming $e = \frac{1}{10}(1 + e_1\cos\eta_{\prime})$, we have

$$B = 9m'^{-1}\delta\mu = 28.83$$

and

$$e_1 = -0.183;$$

i. e.

$$e = \quad 0.1000 - 0.0183\cos\eta_{\prime}.$$

We have, also,

$$3n' - 4n = 0^\circ.0514,$$

$$\gamma = 0.003038;$$

whence, substituting in (32),

$$m' = \frac{1}{4528},$$

and the long period inequality in (34″) becomes

$$-\frac{1}{\gamma}\int n_\eta d\eta = \frac{0.069 \times 57^\circ.3 \sin \eta}{0.003038 \times 4528} = 0^\circ.287 \sin \eta.$$

The terms in θ, 2θ, 4θ, and 5θ give

$$a^2R = -\ 3.23 \cos \theta - 1.31 \cos 2\theta - 1.31 \cos 4\theta - 1.98 \cos 5\theta$$
$$+ (0.38 \cos \theta + 0.31 \cos 2\theta + 0.21 \cos 4\theta + 0.07 \cos 5\theta) \cos \eta,$$
$$Sr = -\ 0.44 \sin \theta + 0.76 \sin 2\theta + 0.39 \sin 4\theta + 0.57 \sin 5\theta$$
$$+ (-\ 0.06 \sin \theta - 0.06 \sin 2\theta - 0.14 \sin 4\theta - 0.17 \sin 5\theta) \cos \eta;$$

whence

$$P_1 = +\ 0.46 + 0.39 \cos \eta, \qquad S_1 = +\ 1.59 + 0.22 \cos \eta,$$
$$P_2 = -\ 4.46 + 0.13 \cos \eta, \qquad S_2 = -\ 1.38 + 0.11 \cos \eta,$$
$$P_4 = -\ 2.47 + 0.15 \cos \eta, \qquad S_4 = -\ 0.35 + 0.12 \cos \eta,$$
$$P_5 = -\ 2.29 + 0.19 \cos \eta; \qquad S_5 = -\ 0.41 + 0.12 \cos \eta.$$

The resulting values of a_i and n_i are

$$a_1 = -\ 0.0002 + 0.0001 \cos \eta, \qquad n_1 = +\ 0.0004 - 0.0002 \cos \eta,$$
$$a_2 = -\ 0.0018 + 0.0001 \cos \eta, \qquad n_2 = +\ 0.0034 - 0.0001 \cos \eta,$$
$$a_3 = \quad 0.0000, \qquad n_3 = -\ 0.0002,$$
$$a_4 = +\ 0.0005 - 0.0001 \cos \eta, \qquad n_4 = -\ 0.0012 + 0.0002 \cos \eta,$$
$$a_5 = +\ 0.0006, \qquad n_5 = -\ 0.0018 + 0.0001 \cos \eta,$$
$$a_6 = +\ 0.0001, \qquad n_6 = -\ 0.0003;$$
$$a_7 = -\ 0.0001;$$

whence, neglecting terms involving g_i and those containing both η and θ,

$$r = a(s_i + \delta s) = as_i - 0.0001 \cos \theta - 0.0022 \cos 2\theta$$
$$+ 0.0006 \cos 4\theta + 0.0007 \cos 5\theta + 0.0001 \cos 6\theta$$
$$- 0.0001 \cos 7\theta.$$

We have, also,

$$w_{\prime} = w_0 + nt - (2e - \tfrac{1}{4}e^3)\sin 3\theta + \tfrac{5}{4}e^2 \sin 6\theta - \tfrac{13}{12}e^3 \sin 9\theta + \tfrac{103}{96}e^4 \sin 12\theta,$$

in which 3θ is the mean anomaly counted from *apo*saturnium, and

$$(Am' + \gamma)(w_{\prime})_\theta = -2e(Am' + \gamma)\sin 3\theta$$
$$= (-0.0006 + 0.0001 \cos \eta_{\prime})\sin 3\theta;$$

whence, neglecting terms involving h_i and those containing both $\eta_{\prime}$ and θ, equation (34″) gives

$$w = w_{\prime} + 0°.29 \sin \eta_{\prime} + 0°.17 \sin \theta + 0°.29 \sin 2\theta - 0°.01 \sin 3\theta$$
$$- 0°.05 \sin 4\theta - 0°.06 \sin 5\theta - 0°.01 \sin 6\theta.$$

14. For $\eta_{\prime} = 90°$, the forces were computed on the same hypothesis as before, and also on the assumption that w contains the additional term,

$$\Delta w = -h_3 \cos 3\theta = -2°.50 \cos 3\theta.$$

The corresponding term in r is, approximately,

$$\Delta r = ag_3 \sin 3\theta = -\tfrac{1}{2}ah_3 \sin 3\theta = -0.0264 \sin 3\theta.$$

For $\eta_{\prime} = 270°$, when $\lambda = 0$, the numerical values of the forces are the same as for $\eta_{\prime} = 90°$; except that the sine terms in a^2R and the cosine terms in Sr change sign.

The values of a^2R and Sr obtained for $\eta_{\prime} = 90°$ are as follows:—

θ	$\Delta w = 0$		$\Delta w = -2^\circ.5 \cos 3\theta$	
	a^2R	Sr	a^2R	Sr
0	$-$ 14.08	$+$ 1.87	$-$ 12.81	$+$ 2.93
2.5	$-$ 12.31	$+$ 3.47	$-$ 10.77	$+$ 3.94
5	$-$ 10.03	$+$ 4.11	$-$ 8.69	$+$ 4.14
7.5	$-$ 8.03	$+$ 4.05	$-$ 6.98	$+$ 3.86
10	$-$ 6.44	$+$ 3.65	$-$ 5.69	$+$ 3.41
15	$-$ 4.39	$+$ 2.67	$-$ 4.07	$+$ 2.50
20	$-$ 3.34	$+$ 1.86	$-$ 3.19	$+$ 1.80
25	$-$ 2.75	$+$ 1.29	$-$ 2.69	$+$ 1.31
30	$-$ 2.38	$+$ 0.89	$-$ 2.38	$+$ 0.97
35	$-$ 2.15	$+$ 0.62	$-$ 2.17	$+$ 0.72
40	$-$ 1.98	$+$ 0.41	$-$ 2.04	$+$ 0.54
45	$-$ 1.88	$+$ 0.25	$-$ 1.95	$+$ 0.39
50	$-$ 1.79	$+$ 0.13	$-$ 1.87	$+$ 0.26
55	$-$ 1.72	$+$ 0.02	$-$ 1.81	$+$ 0.14
60	$-$ 1.63	$-$ 0.10	$-$ 1.73	$-$ 0.01
70	$-$ 1.41	$-$ 0.35	$-$ 1.46	$-$ 0.30
80	$-$ 1.09	$-$ 0.61	$-$ 1.13	$-$ 0.62
90	$-$ 0.71	$-$ 0.83	$-$ 0.70	$-$ 0.86
100	$-$ 0.33	$-$ 0.96	$-$ 0.29	$-$ 1.01
110	$+$ 0.02	$-$ 1.01	$+$ 0.08	$-$ 1.04
120	$+$ 0.35	$-$ 0.98	$+$ 0.41	$-$ 0.95
130	$+$ 0.67	$-$ 0.87	$+$ 0.72	$-$ 0.82
140	$+$ 0.94	$-$ 0.65	$+$ 0.96	$-$ 0.62
150	$+$ 1.14	$-$ 0.40	$+$ 1.13	$-$ 0.40
160	$+$ 1.20	$-$ 0.19	$+$ 1.20	$-$ 0.19
170	$+$ 1.19	$-$ 0.02	$+$ 1.18	$-$ 0.06

θ	$\Delta w = 0$		$\Delta w = -2^\circ.5 \cos 3\theta$	
	a^2R	Sr	a^2R	Sr
180	+ 1.13	+ 0.05	+ 1.13	+ 0.07
190	+ 1.09	+ 0.10	+ 1.09	+ 0.01
200	+ 1.07	+ 0.18	+ 1.07	+ 0.15
210	+ 1.04	+ 0.31	+ 1.05	+ 0.32
220	+ 0.98	+ 0.55	+ 0.97	+ 0.56
230	+ 0.82	+ 0.81	+ 0.78	+ 0.85
240	+ 0.47	+ 1.06	+ 0.42	+ 1.08
250	0.00	+ 1.15	− 0.06	+ 1.13
260	− 0.50	+ 1.01	− 0.55	+ 0.97
270	− 0.90	+ 0.73	− 0.93	+ 0.69
280	− 1.19	+ 0.41	− 1.15	+ 0.41
290	− 1.30	+ 0.21	− 1.23	+ 0.26
300	− 1.36	+ 0.13	− 1.28	+ 0.23
305	− 1.40	+ 0.12	− 1.32	+ 0.22
310	− 1.48	+ 0.11	− 1.40	+ 0.21
315	− 1.60	+ 0.07	− 1.53	+ 0.17
320	− 1.77	− 0.02	− 1.71	+ 0.08
325	− 2.04	− 0.20	− 1.99	− 0.11
330	− 2.40	− 0.50	− 2.39	− 0.43
335	− 2.94	− 0.98	− 2.98	− 0.94
340	− 3.83	− 1.70	− 3.98	− 1.67
345	− 5.49	− 2.70	− 6.02	− 2.71
350	− 8.65	− 3.52	− 9.44	− 3.25
352.5	− 10.89	− 3.35	− 11.71	− 2.66
355	− 13.12	− 2.31	− 13.54	− 1.12
357.5	− 14.43	− 0.35	− 13.98	+ 1.04

This table was completed by interpolation so as to contain values of the forces for every two and a half degrees of θ; whence, considering terms involving multiples of 3θ only, we have for $\Delta\varpi = 0$, $\Delta r = 0$,

$$a^2R = -1.287 - 1.168\cos 3\theta - 0.801\cos 6\theta - 0.489\cos 9\theta - 0.294\cos 12\theta$$
$$+(0.059\sin 3\theta + 0.134\sin 6\theta + 0.131\sin 9\theta + 0.109\sin 12\theta)\sin\eta,$$
$$Sr = 0.654\sin 3\theta + 0.506\sin 6\theta + 0.408\sin 9\theta + 0.214\sin 12\theta$$
$$+ 0.127\sin 15\theta + 0.072\sin 18\theta$$
$$+(0.077 + 0.091\cos 3\theta + 0.089\cos 6\theta + 0.094\cos 9\theta + 0.078\cos 12\theta$$
$$+ 0.064\cos 15\theta + 0.047\cos 18\theta)\sin\eta;$$

whence

$$Q_3 = 0.261\sin\eta,$$
$$Q_6 = 0.200\sin\eta,$$
$$Q_9 = 0.186\sin\eta,$$
$$Q_{12} = 0.141\sin\eta.$$

T_3, T_6, etc., are negligible.

In the same manner, for $\Delta\varpi = -2^\circ.5\cos 3\theta$, $\Delta r = -0.0264\sin 3\theta$,

$$a^2R = -1.259 - 1.113\cos 3\theta - 0.743\cos 6\theta - 0.429\cos 9\theta - 0.250\cos 12\theta$$
$$+(0.110\sin 3\theta + 0.227\sin 6\theta + 0.224\sin 9\theta + 0.180\sin 12\theta)\sin\eta,$$
$$Sr = 0.635\sin 3\theta + 0.468\sin 6\theta + 0.372\sin 9\theta + 0.176\sin 12\theta$$
$$+ 0.093\sin 15\theta + 0.045\sin 18\theta$$
$$+(0.128 + 0.126\cos 3\theta + 0.158\cos 6\theta + 0.159\cos 9\theta + 0.133\cos 12\theta$$
$$+ 0.100\cos 15\theta + 0.074\cos 18\theta)\sin\eta,$$
$$\frac{\sqrt{(1-\nu)}}{\sqrt{a}}\,\frac{n'}{n'-n}\,U = -0.763\cos 3\theta - 0.281\cos 6\theta - 0.126\cos 9\theta - 0.053\cos 12\theta$$
$$- 0.023\cos 15\theta - 0.009\cos 18\theta$$
$$+(0.151\sin 3\theta + 0.095\sin 6\theta + 0.064\sin 9\theta + 0.040\sin 12\theta$$
$$+ 0.024\sin 15\theta + 0.015\sin 18\theta)\sin\eta.$$

Multiplying the latter by

$$2s^{-3} = 2.030 - 0.606\cos 3\theta + 0.090\cos 6\theta - 0.014\cos 9\theta + 0.002\cos 12\theta$$
$$+(0.158\sin 3\theta - 0.008\sin 6\theta)\sin\eta,$$

we have

$$U' = (0.301 \sin 3\theta + 0.077 \sin 6\theta + 0.085 \sin 9\theta + 0.052 \sin 12\theta) \sin \eta;$$

and, adding a^2R,

$$Q_3 = 0.411 \sin \eta,$$
$$Q_6 = 0.304 \sin \eta,$$
$$Q_9 = 0.309 \sin \eta,$$
$$Q_{12} = 0.232 \sin \eta.$$

In the present case, in a first approximation, the values of Q_3, Q_6, etc., may be considered as depending on the value of $\Delta\varpi$ for $\theta = 0$.

The above values give, by means of (24′) and (34″), for $\eta = 90^\circ$,

Assumed.	Computed.
$\Delta\varpi = -2^\circ.5 \cos 3\theta$,	$\Delta\varpi = +0^\circ.287 - 1^\circ.471 \cos 3\theta + 0^\circ.195 \cos 6\theta$
	$-0^\circ.024 \cos 9\theta + 0^\circ.002 \cos 12\theta$,
	$= -1^\circ.011$, for $\theta = 0$,
$\Delta\varpi = 0$,	$\Delta\varpi = +0^\circ.287 - 0^\circ.929 \cos 3\theta + 0^\circ.123 \cos 6\theta$
	$-0^\circ.014 \cos 9\theta + 0^\circ.002 \cos 12\theta$,
	$= -0^\circ.531$, for $\theta = 0$;

whence, by interpolation, $\Delta\varpi$ (assumed) $= \Delta\varpi$ (computed) $= -0^\circ.657$, for $\theta = 0$.

Hence, approximately,

$$Q_3 = 0.300 \sin \eta,$$
$$Q_6 = 0.227 \sin \eta,$$
$$Q_9 = 0.218 \sin \eta,$$
$$Q_{12} = 0.165 \sin \eta,$$

$$\Delta\varpi = 0^\circ.29 - 1^\circ.07 \cos 3\theta + 0^\circ.14 \cos 6\theta - 0^\circ.02 \cos 9\theta,$$
$$\Delta r = \quad - 0.0100 \sin 3\theta + 0.0010 \sin 6\theta - 0.0001 \sin 9\theta.$$

For $\eta = 270^\circ$ and $\lambda = 0$, the values of these inequalities are the same as those for $\eta = 90^\circ$, but with opposite signs.

In a similar manner, a^2R, δr, $\Delta\varpi$, and Δr may be found for other values of η, and the coefficients of the sines and cosines of multiples of θ expressed as periodic functions of that quantity (η) by means of mechanical quadratures.

15. With the values of the co-ordinates thus found more accurate values may be found by a repetition of the process described in the preceding pages; but the solution cannot be considered as even approaching completion until the value of λ is obtained by the comparison of an assumed orbit with observations made near conjunction, and distributed, if possible, over the whole of the nineteen years during which η makes a cycle of 360°.

It may be noted that $(Sr)_0$ as given by equation (40) is nearly proportional to $de^2/d\eta$, and, therefore, is liable to converge slowly, since e^2 converges more slowly than e, and $de^2/d\eta$ converges more slowly than e^2. Hence, the values of $(Sr)_0$ obtained for 90° and 270° are not sufficient to give, approximately even, the coefficient of $\sin \eta$ in that quantity. On the other hand, $-\frac{1}{\gamma}\int n_\eta d\eta$ converges more rapidly than n_η; whence it is probable that the value of that inequality may be derived with considerable accuracy from the forces obtained for $\eta = 0$ and $\eta = 180°$; but, in computing those forces the inequality in e should not be neglected, except in a first approximation.

16. The method here suggested may be readily extended to cases in which the disturbed and disturbing bodies do not move in coplanar orbits. The asteroids furnish quite a number of interesting cases of mean motions nearly commensurate with that of Jupiter. As examples may be mentioned (153) Hilda and (190) Ismene, whose mean motions are each approximately three halves that of Jupiter.

SOLUTIONS OF EXERCISES.

171

A homogeneous heavy rod is hung from a fixed point by elastic threads of given length fastened at its extremities. Find the position of equilibrium.

[*W. M. Thornton.*]

SOLUTION.

Assuming the rod to be of uniform section, the position of equilibrium is the same as if W, the weight of the rod, was concentrated at its middle point, through which passes the vertical through the point of suspension.

Let l and l' be the given lengths of the strings, and μ, μ' such weights as would stretch them to double their original lengths. Let dl and dl' be their respective elongations, and θ the angle between the strings in the position of equilibrium.

T and T' being the tensions in the strings, we have from statical relations

$$T^2 + T'^2 + 2TT' \cos\theta = W^2; \tag{1}$$

also
$$T/T' = (l + dl)/(l + dl').$$

By Hooke's law,
$$T = \mu dl/l; \quad T' = \mu' dl'/l'. \tag{2}$$

Hence
$$T/T' = \mu l' dl/\mu' l dl' = (l + dl)/(l' + dl'). \tag{3}$$

From (1) and (2) result

$$\left(\frac{\mu l' dl}{\mu' l dl'}\right)^2 + 1 + 2\,\frac{\mu l dl'}{\mu' l' dl} \cos\theta = \left(\frac{Wl'}{\mu' dl'}\right)^2;$$

and from the geometry of the figure we get

$$\left(\frac{l + dl}{l' + dl'}\right)^2 + 1 + 2\,\frac{l + dl}{l' + dl'} \cos\theta = \left(\frac{2x}{l' + dl'}\right)^2,$$

where x is the distance of the middle of the rod below the point of suspension.

These two equations give, by virtue of relation (3),

$$dl' = \frac{Wl'^2}{2\mu' x - Wl'}, \qquad \text{and also} \qquad dl = \frac{Wl^2}{2\mu x - Wl}.$$

Therefore the stretched lengths of the strings are

$$\frac{l'}{1 - \dfrac{Wl'}{2\mu' x}} \quad \text{and} \quad \frac{l}{1 - \dfrac{Wl}{2\mu x}},$$

where x is to be found from the geometrical relation

$$\left(\frac{l'}{1 - \dfrac{Wl'}{2\mu' x}}\right)^2 + \left(\frac{l}{1 - \dfrac{Wl}{2\mu x}}\right)^2 = 2(x^2 + a^2),$$

in which a is half the length of the given rod. This equation for x is of the sixth degree, which is best solved when μ and μ' are large, and therefore the elongations $Wl/2\mu$ and $Wl'/2\mu'$ are small compared with x, by using the median of the unstretched triangle as an approximate value of x and substituting successively in the first member of the equation. [*W. H. Echols.*]

252

$f(x)$ and $\varphi(e^x)$ being rational algebraic functions of x and e^x, respectively, show that

$$\int f(x)\,\varphi(e^x)\,dx$$

depends upon integrals of three forms,

$$\int \frac{du}{\log u}, \quad \int \theta^m \tan\theta\, d\theta, \quad \int \frac{du}{(u+c)^r \log u}.$$

[*R. A. Harris.*]

SOLUTION.

There are, evidently, four forms to be considered; viz.,

$$\int x^m e^{nx} dx, \quad \int \frac{e^{nx}}{(x+a)^m}\, dx, \quad \int \frac{x^m}{(e^x + b)^n}\, dx, \quad \int \frac{dx}{(x+a)^m (e^x + b)^n}.$$

1. The first of these integrals can be found by successive integrations by parts.

2. The second, when treated in like manner, will be found to depend upon

$$\int \frac{e^{nx}}{x+a}\, dx.$$

This, in turn, depends upon the well-known transcendent

$$\int \frac{du}{\log u}.$$

3. Let us transform the expansion $\frac{x^m}{(e^x+b)^n}dx$ by letting $z=e^x$; then its integration may be made dependent upon integrals of the form

$$\int\frac{(\log z)^m}{z+b}dz,$$

as follows from considering the partial fractions into which

$$\frac{(\log z)^m}{z(z+b)^n}dz\left(=\frac{x^m}{(e^x+b)^n}dx\right)$$

may be decomposed. Next, let $u=z/b$; then the above integral will depend upon

$$\int\frac{(\log u)^m}{u+1}du.$$

If we let $\cos\theta+i\sin\theta=u^{\frac{1}{2}}$ it is not difficult to show that this integral depends upon

$$\int\theta^m\tan\theta\,d\theta.$$

When x and b are real and the latter is essentially negative, we should write $u=z/-b$, $\cos\theta-i\sin\theta=u^{\frac{1}{2}}$, thus making the integral in question depend upon

$$\int\theta^m\cot\theta\,d\theta,$$

which is, obviously, reducible to integrals of the form just written. It may be observed that when mod. θ lies between 0 and 1, mod. u lies between $1/e^2$ and e^2.

4. If

$$\frac{dx}{(x+a)^m(e^x+b)^u}$$

be integrated by parts $m-1$ times, the integration will be seen to depend upon integrals of the form

$$\int\frac{e^{qx}}{(e^x+b)^r}\cdot\frac{dx}{x+a},$$

where $q \overline{\overline{<}}\, m-1$, $r \overline{\overline{<}}\, m+u-1$. This, in turn, depends upon

$$\int\frac{du}{(u+c)^r\log u}.$$

[*R. A. Harris.*]

EXERCISES.

271

If

$$\Delta = \begin{vmatrix} a_1 & b_1 & c_1 & \dots & h_1 \\ a_2 & b_2 & c_2 & \dots & h_2 \\ a_3 & b_3 & c_3 & \dots & h_3 \\ \dots & \dots & \dots & \dots & \dots \\ a_n & b_n & c_n & \dots & h_n \end{vmatrix}, \text{ and } D = \begin{vmatrix} \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} & \begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix} & \dots & \begin{vmatrix} a_1 & h_1 \\ a_2 & h_2 \end{vmatrix} \\ \begin{vmatrix} a_1 & b_1 \\ a_3 & b_3 \end{vmatrix} & \begin{vmatrix} a_1 & c_1 \\ a_3 & c_3 \end{vmatrix} & \dots & \begin{vmatrix} a_1 & h_1 \\ a_3 & h_3 \end{vmatrix} \\ \dots & \dots & \dots & \dots \\ \begin{vmatrix} a_1 & b_1 \\ a_n & b_n \end{vmatrix} & \begin{vmatrix} a_1 & c_1 \\ a_n & c_n \end{vmatrix} & \dots & \begin{vmatrix} a_1 & h_1 \\ a_n & h_n \end{vmatrix} \end{vmatrix};$$

then will $D = a_1^{n-2}\Delta.$ [*T. M. Blakslee.*]

272

Prove geometrically or by circular co-ordinates (or in any other way), that if the sides of a triangle of given shape pass through fixed points,

1. Any line connected with the moving triangle passes through a fixed point;
2. Any connected circle envelopes a nodal bicircular quartic;
3. All such quartics have a common node. [*Frank Morley.*]

273*

The asymptotes of a curve all meet at P. Show that P is the mean centre of the feet of the normals from it to the curve. [*Frank Morley.*]

274

At what point on the surface of a sphere will an observer see the reflection of a distant light? [*L. G. Carpenter.*]

275

Investigate a formula for the shortening of the day at a place of latitude φ by a chain of mountains of angular altitude h and azimuth A.

[*L. G. Carpenter.*]

276

Show that the attraction of a finite mass on one of its points is finite.

[*A. Hall.*]

*See a magnificent paper by Humbert, *Journal de Math.*, 4th series, tome iii, pp. 327–404, where by a novel use of Abel's Theorem one's ideas on the properties of algebraic plane curves receive a decided generalization. The above exercise is a special variation on a thème there given (p. 362).

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